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Loop Integrals in Three Outstanding Gauges: Feynman, Light-Cone, and Coulomb

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We apply the negative dimensional integration method (NDIM) to three outstanding gauges: Feynman, light-cone, and Coulomb gauges. Our aim is to show that NDIM is a very suitable technique to deal with loop integrals, regardless of which gauge choice that originated them. In the Feynman gauge we perform scalar two-loop four-point massless integrals; in the light-cone gauge we calculate scalar two-loop integrals contributing to two-point functions without any kind of prescriptions, since NDIM can abandon such devices—this calculation is the first test of our prescriptionless method beyond one-loop order; and finally, for the Coulomb gauge we consider a four-propagator massless loop integral, in the split-dimensional regularization context.

Key Words: quantum field theory; negative dimensional integration; dimensional regularization; noncovariant gauges: light-cone and Coulomb.

1. INTRODUCTION

The perturbative approach to quantum field theory in any gauge deals with Feynman diagrams, which can be expressed as D-dimensional integrals. The success of such an approach is readily apparent from the comparison between the theoretical and experimental values for the electron's anomalous magnetic moment, that is, by the quantity $a = \frac{1}{2}(g-2)$ measure for the electron (see for instance [1]):

$$\begin{aligned} a_{The} &= 1159652201.2(2.1)(27.1) \times 10^{-12} \\ a_{Exp} &= 1159652188.4(4.3) \times 10^{-12}. \end{aligned} \tag{1}$$

This is the best motivation to study quantum field theory, since no physical theory has given such accuracy in any measurement. In other words, it is the very best we have up to now.

In Section 2 we discuss sample two-loop four-point functions, namely, on-shell double boxes with five and six massless covariant propagators; Section 3 is devoted to

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noncovariant gauges, namely the light-cone and the Coulomb ones. The integrals we study for the former noncovariant gauge have seven propagators (two-loops) and the latter is a one-loop having four propagators, which is complicated by the necessity to use split-dimensional regularization (SDR). In the final Section 4, we present our concluding remarks.

2. FEYNMAN GAUGE: SCALAR TWO-LOOP FOUR-POINT MASSLESS INTEGRALS

Of course, covariant gauges are the most popular, in what we could call the "gauge market" [2]. Several methods were and are still developed to evaluate complicated Feynman loop integrals, from the perspective of both analytic and numerical results [1, 3, 4], all in the context of dimensional regularization [5].

Our work is concerned with the application of the negative-dimensional integration method (NDIM). It is a technique which can be applied to any gauge, covariant or noncovariant alike. The results are always expressed as hypergeometric series which have definite regions of convergence allowing us to study the referred diagrams or processes in specific kinematical regions of external momenta and/or masses.

However, as life teaches us all, not everything in the NDIM method is peaches and cream; it has its drawback: the amazing number of series—in the case where one is considering massless diagrams—which must be summed. When such sums are of Gaussian type, it is quite easy to write a small computer program that can do the job algebraically. However, when the series are of superior order, ${}_{p+1}F_p$, for $p \geq 2$, there are no known formulas which can reduce it to a product of gamma functions for any value of its parameters. (The few exceptions as contemplated by theorems, such as Saalschutzian's, when one of the numerator parameters is negative, generally do not apply to our cases.) Despite this technical problem, NDIM has proven to be an excellent method [6–9] for computing loop integrals.

In the task of calculating perturbative Feynman diagrams, a question that arises quite often is which is more difficult to handle, graphs with more loops or graphs with more legs? In our point of view, i.e., in the context of NDIM, the greater the number of loops the heftier the calculations needed to solve the problem. We will consider in this section a diagram which has both (great number of legs and loops, four and two, respectively): a scalar two-loop double-box integral where all the particles are massless and the external legs are on-shell.

2.1. *Double Box with Five and Six Propagators*

Let us consider the diagram of Fig. 1. Consider as the generating functional for our negative-dimensional integral the Gaussian one, where all external legs are on-shell,

$$G_b = \int d^D q\, d^D r\ \exp\left[-\alpha q^2 - \beta(q-p)^2 - \gamma(q-p-p')^2 - \theta(q-r-p_1)^2 - \phi r^2 - \omega(q-r)^2\right], \tag{2}$$

$$= \left(\frac{\pi^2}{\Lambda}\right)^{D/2} \exp\left[\frac{1}{\Lambda}(-\gamma\phi\omega s - \beta\theta\phi t)\right], \tag{3}$$

where (s, t) are the usual Mandelstam variables and we use $s + t + u = \Sigma m_i^2 = 0$. Observe that in the particular case where $\alpha = 0$ we recover the Gaussian integral for the diagram of Fig. 2. We also define $\Lambda = \alpha\theta + \alpha\phi + \alpha\omega + \beta\theta + \beta\phi + \beta\omega + \gamma\theta + \gamma\phi + \gamma\omega + \phi\omega + \theta\phi$.

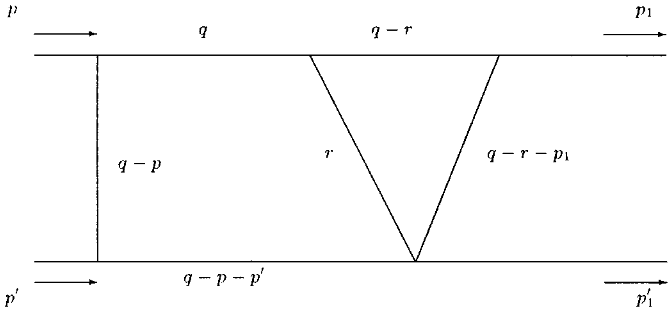

FIG. 1. Double box with six propagators.

The usual technique reveals that there are thirteen sums and seven equations. From the combinatorics one can solve such constraints in 1716 different ways. Of course several systems have no solution—not even in the homogeneous case—and from our previous works we know that some results are n-fold degenerate while others are related by analytic continuation. The result for the integral in question is

$$\mathcal{BOX} = \int d^D q\, d^D r\, (q^2)^i (q-p)^{2j} (q-p-p')^{2k} (q-r-p_1)^{2l} (r^2)^m (q-r)^{2n} \quad (4)$$

$$= (-\pi)^D i!\, j!\, k!\, l!\, m!\, n!\, \Gamma(1-\sigma_b - D/2) \sum_{\text{all}=0}^{\infty} \frac{s^{X_1} t^{X_2}}{X_1! X_2! Y_1! \ldots Y_9! Z_1! Z_2!} \delta, \quad (5)$$

where “all” means $\{X_1, X_2, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9, Z_1, Z_2\}$ and

$$\mathcal{BOX} = \mathcal{BOX}(i, j, k, l, m, n)$$

must be understood and **delta** represents the system of constraints. The above expression can be expressed, in principle, as a seven-fold hypergeometric series, for which there are three possibilities,

$$\mathcal{F}(\ldots|z, z^{-1}, 1), \quad \mathcal{F}(\ldots|z, 1), \quad \text{and} \quad \mathcal{F}(\ldots|z^{-1}, 1), \quad (6)$$

where $z = -s/t$. Some series with unit argument, if they are Gaussian can be summed up.

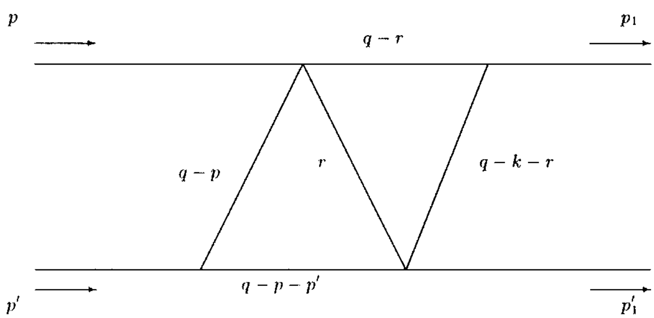

FIG. 2. Double box with five propagators.

TABLE I
Parameters of Hypergeometric Functions $_3F_2$ Representing Box Integrals

Parameters	$_3F_2(\{1\}\|z)$	$_3F_2(\{2\}\|z)$
a	$-k$	$-k$
b	$-n$	$-n$
c	$-\sigma_b$	$-\sigma_b'$
e	$1+j-\sigma_b$	$1+j-\sigma_b'$
f	$1+l-\sigma_b$	$1+l-\sigma_b'$

However, a hypergeometric function is meaningful only if the series which defines it is convergent. It is easy to see that the first possibility above, that is, $\mathcal{F}(\ldots|z, z^{-1}, 1)$, cannot satisfy this requirement of convergence and therefore we disregard it.

Among the 624 total solutions of the system of constraints we look for the simplest solution, namely, the one in which we can sum the greatest number of series. It is not difficult to find it using computer facilities, and we get

$$\mathcal{BOX}^{AC}(i,j,k,l,m,n) = f_1(i,j,k,l,m,n)\,{}_3F_2(\{1\}|z), \tag{7}$$

where the five parameters are quoted in the Table I, $\sigma_b = i+j+k+l+m+n+D$, and

$$\begin{aligned} f_1(i,j,k,l,m,n) = \pi^D (t^2)^{\sigma_b}&(-j|\sigma_b)(-l|\sigma_b)(\sigma_b + D/2| - 2\sigma_b - D/2)\\ &\times(-m|l+m+n+D/2)(i+j+k+m+D|-m-D/2)\\ &\times(l+m+n+D|-l-n-D/2). \end{aligned}\tag{8}$$

Besides this one, we can have Appel's, Lauricella's, and even more complicated hypergeometric functions. Moreover,

$$(x|y) \equiv (x)_y = \frac{\Gamma(x+y)}{\Gamma(x)}.$$

If we remember that the final result should be the sum of linearly independent series [7, 10], we can rightfully ask if one is not missing two other $_3F_2$ functions. According to Luke and Slater [12], the differential equation for $_pF_q$ has p linearly independent solutions, so we would in principle expect in our case above a sum of three terms. On the other hand, according to Nørlund [13], if the difference between an upper parameter and a lower one is an integer number (which is our case above—see Table I), then some series do no exist—we used this theorem in [7]. So, Eq. (7) is the final result for the referred integral in the region where $|z| < 1$. The expression for the same graph outside this region can be obtained making the substitutions

$$s \leftrightarrow t, \quad j \leftrightarrow k, \quad l \leftrightarrow n \tag{9}$$

so that we have another $_3F_2$ hypergeometric function as the result for $|z| > 1$.

Another solution for the Feynman integral can be written as a triple hypergeometric series,

$$\mathcal{BOX}_3 = \pi^D t^j s^{\sigma_b - j} f_3 \sum_{Y_i=0}^{\infty} \frac{(-j|Y_{456})(m+D/2|Y_{46})(l+n+D/2|Y_5)}{Y_4!Y_5!Y_6!(1+\sigma_b - j|Y_{456})} \times \frac{(i+k+l+m+D|Y_{45}+2Y_6)(i+l+m+n+D|Y_{456})z^{Y_{456}}}{(1-j+l|Y_{56})(i+j+k+m+D|Y_{46})(l+m+n+D|Y_{456})} + (j \leftrightarrow l), \tag{10}$$

where $Y_{ij\cdots} \equiv Y_i + Y_j + \cdots$ and

$$f_3 = (-l|j)(l+m+n+D|i)(\sigma_b + D/2|-j-n-D/2)(-n|l+2n+D/2) \times (i+j+k+m+D|-m-D/2)(-k|j+k-\sigma_b)(-m|2m+D/2). \tag{11}$$

Obviously, the above series converges if $|z| < 1$, in addition to other possible conditions on z. So there is an overlapping between the regions of convergence of $\mathcal{BOX}$ and $\mathcal{BOX}_3$, so that there is an analytic continuation formula which relates both. As far as we know, in the pertinent literature there is not a known formula relating triple hypergeometric series to simple ones.

We also have solutions written in terms of a four-fold series, i.e.,

$$\mathcal{BOX}_4 = \pi^D t^l s^{\sigma_b - l} f_4 \sum_{Y_i=0}^{\infty} \frac{(-l|Y_{1247})(m+D/2|Y_{147})(-i|Y_1)}{Y_1!Y_4!Y_7!Y_2!(1+\sigma_b - l|Y_{1247})(1+j-l|Y_{127})} \times \frac{(i+j+k+D/2|Y_2)(i+j+m+n+D|Y_{124}+2Y_7)z^{Y_{1247}}}{(l+m+n+D|Y_{147})} + (j \leftrightarrow l), \tag{12}$$

where

$$f_4 = (-j|l)(l+m+n+D|i+j-l)(\sigma_b + D/2|-k-l-D/2)(-m|2m+D/2) \times (-k|k+l-\sigma_b)(-n|l+2n+D/2). \tag{13}$$

Observe that the two previous results are singular when $j - l =$ integer, since we have $\Gamma(j-l)$ or $\Gamma(l-j)$ in the numerator. However, this singularity cancels out if one considers exponents of propagators in the analytic regularization context, i.e., if we introduce [7, 11] for instance $j = -1 + \delta$ and then expand the whole expression around $\delta = 0$. Proceeding in this way the pole in δ cancels out.

Note that we have here reduction formulas which transform a hypergeometric function defined by triple and four-fold series in a simpler function defined by a unique sum. These formulas are not in the textbooks on the subject. *It is an original result.*

2.1.1. Double box with five propagators. The graph of Fig. 2, is a special case of the previous one. In the Gaussian integral α must be zero, so in the final result we must merely take $i = 0$,

$$\mathcal{BOX}^{AC}(0, j, k, l, m, n) = f(0, j, k, l, m, n)_3F_2(\{2\}|z), \tag{14}$$

where the parameters are listed in Table I and we define $\sigma'_b = j + k + l + m + n + D$ and

$$f(0, j, k, l, m, n) = \pi^D (t^2)^{\sigma'_b} (-j|\sigma'_b)(-l|\sigma'_b)(\sigma'_b + D/2| - 2\sigma'_b - D/2) \times (-m|l + m + n + D/2)(j + k + m + D| - m - D/2) \times (l + m + n + D| - l - n - D/2). \quad (15)$$

We can proceed with the same substitutions as in (9) to obtain the result outside the region $|z| < 1$.

Finally, when all the exponents are equal to minus one, the ${}_3F_2$ collapses into a Gaussian ${}_2F_1$ which can be written as an elementary function.

The results for $\mathcal{BOX}_3$ and $\mathcal{BOX}_4$ for this special case ($i = 0$) are hypergeometric series representations for the integral in question. We note that for $\mathcal{BOX}_4$, we had a Pochhammer $(-i|Y_1)$, which for $i = 0$ means automatically $Y_1 = 0$, so that the four-fold sum now becomes a triple sum, say $\mathcal{BOX}'_3$. But this is exactly equal to the previous BOX_3, which means the result is degenerate and the final result for the Feynman integral is the triple series representation for $\mathcal{BOX}_3(0, j, k, l, m, n)$.

3. NONCOVARIANT GAUGES: LIGHT-CONE AND COULOMB

Recently there have been many works on noncovariant gauges, namely, light-cone [14, 15], Coulomb [16], and radial and axial gauges [17]. Despite the fact that they are not so popular as covariant ones, they have some important features which can help our study and understanding of certain physical problems.

The light-cone gauge, as far as we know, is the only one where certain supersymmetric theories can be shown to be UV finite and that possesses a local Nicolai map [18]. Moreover, ghosts decouple from physical particles and we are left with a reduced number of diagrams to work with. On the other hand, the price to pay seemed to be high, since the gauge boson propagator did generate spurious poles in physical amplitudes. This problem was overcome when Mandelstam and Leibbrandt [19] introduced *ad hoc* prescriptions to treat them (there are also other causal prescriptions which can be implemented, proposed by Pimentel and Suzuki [20], known as causal Cauchy principal value prescription.) However, the famous ML-prescription necessarily forces one to use partial fractioning tricks and integration over components, which make the calculations rather involved [21].

The negative-dimensional approach can avoid all devices such as the use of prescriptions and partial fractionings and provide physically acceptable results, i.e., causality preserving results. The calculation we will present is the very first test beyond one-loop order without invoking ML-prescription. This very fact led us to call NDIM a prescriptionless method [8]. Furthermore, integration over components and partial fractioning tricks can be completely abandoned, as can parametric integrals. The important point to note [8] is that the dual light-like 4-vector n^*_μ is necessary in order to span the needed four-dimensional space [18, 22]. Without this dual light-like 4-vector, that is, a theory with only one-degree violation of covariance, the calculations lead to unphysical results (i.e., they break causality) [6].

The second noncovariant gauge we deal with in this paper is the Coulomb gauge. The potential between quarks and studies on confinement are easily performed in this gauge

[15, 16]. In addition, the ghost propagator has no pole in this gauge! As in the light-cone gauge, a theory constrained in the Coulomb gauge also has problems with the boson propagator. In the former, loop integrals generated aditional poles; in the latter, such integrals are not even defined [23] since they have the form

$$\int \frac{dq_4 d^3\mathbf{q}}{\mathbf{q}^2}. \tag{16}$$

Such objects are called energy integrals. Doust and Taylor [23] presented a solution for this problem in a form of a interpolating gauge (between Feynman and Coulomb). Leibbrandt and co-workers [24] presented a different approach, a procedure they called *split-dimensional regularization* (SDR), which introduces two regulating parameters, one for the energy component and another for the 3-momentum one. So, the measure in SDR becomes

$$d^D q = dq_4 d^{D-1}\mathbf{q} \overset{sdr}{\rightarrow} d^D q = d^\rho q_4 d^\omega \mathbf{q} \tag{17}$$

in Euclidean space.

Here we demonstrate that NDIM can also deal with Coulomb gauge loop integrals, as long as we make use of SDR. In this work we propose to apply NDIM to scalar integrals with four massless propagators. Our results are given in terms of hypergeometric series involving external momenta, exponents of propagators, and regulating parameters ω and ρ.

3.1. *The Light-Cone Gauge*

So far, we have tested our NDIM for integrals pertaining to the one-loop class. Now we apply this technology to some massless two-loop integrals. Let us consider an integral studied by Leibbrandt and Nyeo [21], since they did not present the full result for it:

$$C_3 = \int d^D q d^D k \frac{k^2}{q^2(q-k)^2(k-p)^2(k\cdot n)(q\cdot n)}. \tag{18}$$

In their calculation the ML-prescription must be understood. However, in the NDIM context the key point is to introduce the dual vector n^*_μ to span the needed space [8, 18, 22]. If we do not consider it, our result will violate causality, giving the Cauchy principal value of the integral in question, as we concluded in [6].

The advantage of NDIM over other methods is that it can consider lots of integrals in a single calculation. Our aim to is perform

$$\mathcal{N} = \int d^D q d^D k_1 \left(k_1^2\right)^i (q^2)^j (q-k_1)^{2k} (k_1-p)^{2l} (k_1\cdot n)^m (q\cdot n)^s (k_1\cdot n^*)^r. \tag{19}$$

We will carry out this integral and then present results for special cases, including Leibbrandt and Nyeo's C_3, where $i=-1$, $r=0$, and the other exponents equal minus one. Observe that the integral must be considered as a function of external momentum, exponents of propagators, and dimension:

$$\mathcal{N} = \mathcal{N}(i, j, k, l, m, r, s; P, D). \tag{20}$$

Here P represents $(p^2, p^+, p^-, \frac{1}{2}(n\cdot n^*))$, and we adopt the usual notation of light-cone gauge [2].

Our starting point is the generating functional for our negative-dimensional integrals,

$$G_N = \int d^Dq d^Dk\ \exp[-\alpha k^2 - \beta q^2 - \gamma(q-k)^2 - \theta(k-p)^2 - \phi(k\cdot n) - \omega(q\cdot n) - \eta(k\cdot n^*)], \tag{21}$$

which after a little bit of algebra allows momentum integration, which yields

$$G_N = \left(\frac{\pi^2}{\lambda}\right)^{D/2} \exp\left\{\frac{1}{\lambda}\left[-g_1 p^2 - g_2(p\cdot n) - g_3(p\cdot n^*) + g_4\left(\frac{1}{2}n\cdot n^*\right)\right]\right\}, \tag{22}$$

where

$$g_1 = (\alpha\beta + \alpha\gamma + \beta\gamma)\theta, \quad g_2 = (\beta\phi + \gamma\omega + \gamma\phi)\theta, \quad g_3 = (\beta+\gamma)\eta\theta, \quad g_4 = \eta\frac{g_2}{\theta},$$

and $\lambda = \alpha\beta + \alpha\gamma + \beta\gamma + \beta\theta + \gamma\theta$.

Taylor expanding the exponentials we obtain

$$\mathcal{N} = (-\pi^D) i!\,j!\,k!\,l!\,m!\,r!\,s!\,\Gamma(1-\sigma_n - D/2)\sum_{\text{all}=0}^{\infty} \frac{\delta}{X_1!\ldots X_8! Y_1! Y_2! Y_3!} \times \frac{(p^2)^{X_{123}}(p^+)^{X_{456}}(p^-)^{X_{78}}}{Z_1!\ldots Z_5!}\left(-\frac{n\cdot n^*}{2}\right)^{Y_{123}}, \tag{23}$$

where $\sigma_n = i+j+k+l+m+r+s+D$ and δ represents the system of constraints (8×16) for the negative-dimensional integral. At the end of the day we have 12,870 possible solutions for such a system! Most of them, 9142, have no solution, while 3728 present solutions which can be written as hypergeometric series. Of course several of these will provide the same series representation; these solutions we call degenerate.

We present a result for the referred integral as a double hypergeometric series,

$$\mathcal{N} = \pi^D f_n P_n \sum_{Z_j=0}^{\infty} \frac{(\sigma_n + D/2|Z_{45})(i+j+k+m+s+D|Z_{45})(D/2+k|Z_4)}{Z_4! Z_5! (1+i+j+k+\sigma_n+D|Z_{45})(j+k+s+D|Z_{45})} \times \frac{(j+s+D/2|Z_5)(i+j+k+r+D|Z_{45})}{(1+i+j+k+m+r+s+D|Z_{45})}\left(\frac{p^2 n\cdot n^*}{2p^+p^-}\right)^{Z_{45}}, \tag{24}$$

where

$$f_n = (-m|-s)(-i-j-k-D/2|-\sigma_n - D/2)(j+k+s+D|i-s+r) \times \frac{(-l|k+l+D/2)(-k|-j-D/2)(-m|j+m+s+D/2)}{(1+r|-i-j-k-m-r-s-D)} \times(-j|-i-k-m-r-s-D) \tag{25}$$

are the Pochhammer symbols and

$$P_n = (p^2)^{\sigma_n+i+j+k+D}(p^+)^{l+m+s-\sigma_n}(p^-)^{l+r-\sigma_n}\left(\frac{n\cdot n^*}{2}\right)^{\sigma_n - l}. \tag{26}$$

Now we can consider the special case ($i = 1$, $j = k = l = m = s = -1$, $r = 0$), studied in [21],

$$\mathcal{N}_{SC} = \pi^D \frac{\Gamma(5-2D)\Gamma(D-1)\Gamma(D/2-1)\Gamma(2-D/2)\Gamma(D/2-2)}{\Gamma(1-D/2)\Gamma(D-3)} (p^2)^{2D-5} \times (p^+)^{1-D}(p^-)^{3-D}\left(\frac{n\cdot n^*}{2}\right)^{D-3} \sum_{Z_4,Z_5=0}^{\infty} \frac{(3D/2-4|Z_{45})}{Z_4!Z_5!(2D-4|Z_{45})} \times \frac{(D/2-1|Z_4)(D/2-2|Z_5)(D-1|Z_{45})}{(D-2|Z_{45})}\left(\frac{p^2 n\cdot n^*}{p^+p^-}\right)^{Z_{45}}. \tag{27}$$

Observe that it exibits a double pole, as stated by Leibbrandt and Nyeo [21].

3.2. *The Coulomb Gauge*

We will present the full calculation of an integral which has four propagators,

$$J(i, j, k, m) = \int d^Dq (q^2)^i (q-p)^{2j} \mathbf{q}^{2k} (\mathbf{q}+\mathbf{p})^{2m}, \tag{28}$$

where to regulate the possible divergences originated by the energy component, SDR must be understood, namely,

$$d^Dq = d^\rho q_4 d^\omega \mathbf{q}, \tag{29}$$

where $D = \rho + \omega$.

The generating functional for our negative-dimensional integrals is the Gaussian-like integral

$$G_c = \int d^Dq \exp[-\alpha q^2 - \beta(q+p)^2 - \gamma \mathbf{q}^2 - \theta(\mathbf{q}+\mathbf{p})^2], \tag{30}$$

which can be easily integrated to give

$$G_c = \frac{\pi^{D/2}}{\lambda_1^{\rho/2}\lambda_2^{\omega/2}} \exp\left(-\frac{\alpha\beta}{\lambda_1} p_4^2\right) \exp\left[-\frac{(\alpha+\gamma)(\beta+\theta)}{\lambda_2}\mathbf{p}^2\right]. \tag{31}$$

There are results given by double, triple, four-fold, and five-fold hypergeometric series in the variable $\mathbf{p}^2/p_4^2$ or its inverse.

We will present two such hypergeometric series representations. The first one is a four-fold series,

$$J_4(i, j, k, m) = C_4(i, j, k, m) \sum_{X_i=0}^{\infty} \frac{(-i|X_{1234})(j+m+D/2|X_{34})}{X_1!X_2!X_3!X_4!} \left(\frac{\mathbf{p}^2}{p_4^2}\right)^{X_{1234}} \times \frac{(-1)^{X_3}(1+j+m+\rho/2|X_3 - X_{12})(-m|X_2)}{(1+j+k+m+D/2|X_{34})} \times \frac{(-j-\rho/2|X_1 - X_3)(k+\omega/2|X_{124})}{(1-i-\rho/2|X_{124})} + (i \leftrightarrow j, k \leftrightarrow m), \tag{32}$$

where

$$C_4(i,j,k,m) = \pi^{D/2}\left(p_4^2\right)^i (\mathbf{p}^2)^{\sigma_c - i}(-j|-\rho/2)(-j-m-\rho/2|-k-\omega/2)$$
$$\times(-k|2k+\omega/2)(j+k+m+D/2+\omega/2|-k-\omega/2), \qquad (33)$$

with $\sigma_c = i+j+k+m+D/2$.

The second is a five-fold hypergeometric series,

$$J_5(i,j,k,m) = C_5(i,j,k,m)\sum_{X_i=0}^{\infty}\frac{(-i-j-\rho/2|2X_1+X_{2345})}{X_1!X_2!X_3!X_4!X_5!}$$
$$\times\left(\frac{\mathbf{p}^2}{p_4^2}\right)^{2X_1+X_{2345}}\frac{(-1)^{X_5}(m+\omega/2|X_{1345})}{(1+k+m+\omega/2|X_{145})}$$
$$\times\frac{(k+\omega/2|X_{1245})}{(1-j-\rho/2|X_{135})(1-i-\rho/2|X_{124})}, \qquad (34)$$

where

$$C_5(i,j,k,m) = \pi^{D/2}\left(p_4^2\right)^{i+j+\rho/2}(\mathbf{p}^2)^{k+m+\omega/2}(-i|i+j+\rho/2)(-j|i+j+\rho/2)$$
$$\times(-k|k+m+\omega/2)(-m|k+m+\omega/2)(i+j+\rho|-\sigma_c-\rho/2)$$
$$\times(k+m+\omega|-\sigma_c-\omega/2). \qquad (35)$$

Observe that the above result is also symmetric in $(i \leftrightarrow j, k \leftrightarrow m)$, which means in the loop integral, $q^\mu \to q^\mu + p^\mu$.

Another important point to observe is that the final result must be a sum of linearly independent hypergeometric series [6, 7]. The above fivefold series, J_5, appears only once whereas J_4 is degenerate since several systems give its two hypergeometric functions. This must never be forgotten if one wants to apply NDIM to more complicated diagrams which can in principle generate even more involved hypergeometric series representations.

Moreover, the above expressions, J_4 and J_5, are related by direct analytic continuation, since both are convergent for $|\mathbf{p}^2/p_4^2| < 1$. When one is considering simple hypergeometric function, several formulas are known; on the other hand, for rather complicated hypergeometric series, such as the ones we obtained with four- and fivefold series, there are very few such formulas. NDIM can fill this gap, since it is the only method which provides hypergeometric series representations for Feynman loop integrals, in different kinematical regions, and related by analytic continuation either directly or indirectly.

The above hypergeometric series (only the series, not the factors!), J_4 and J_5, can be written as generalized hypergeometric functions [12] of four and five variables,

$$\mathcal{F}^{6:0;0;0;0}_{2:0;0;0;0}\left[\begin{matrix}(-i:1,1,1,1),(j+m+D/2:0,0,1,1)(1+j+m+\rho/2:-1,-1,1,0)\\(1+j+k+m+D/2:0,0,1,1)\end{matrix}\right.$$
$$\left.\begin{matrix}(-m:0,1,0,0)\\(1-i-\rho/2:1,1,0,1)\end{matrix}\middle|\,x,x,-x,x,\right] \qquad (36)$$

and

$$\mathcal{F}^{3:0;0;0;0;0}_{3:0;0;0;0;0}\left[\begin{array}{c}(-i-j-\rho/2:2,1,1,1,1),(m+\omega/2:1,0,1,1,1)\\(1+k+m+\omega/2:1,0,0,1,1),(1-j-\rho/2:1,0,1,0,1)\end{array}\right.$$
$$\left.\begin{array}{c}(k+\omega/2:1,1,0,1,1)\\(1-i-\rho/2:1,1,0,1,0)\end{array}\right| x^2,x,x,x,-x\Bigg], \qquad (37)$$

where $x=\mathbf{p}^2/p_4^2$.

4. CONCLUSION

The technique of Feynman parametrization can of course be used to perform loop integrals in different gauges but it is very difficult to perform parametric integrals for arbitrary exponents of propagators. This is not so with NDIM, for carrying loop integrals out with particular exponents is as easy as dealing with arbitrary ones—besides, one can come across singularities which depend on them and not on dimension D. This fact is very important when studying light-cone gauge Feynman integrals, because one could have to handle products such as $(q^+)^a[(q-p)^+]^b$, with a and b being negative. NDIM can calculate all of them simultaneously, but if one chooses partial fractioning tricks then he/she will be forced to carry out each integral separately. Besides usual covariant integrals and the trickier light-cone gauge ones, NDIM was probed into the Coulomb gauge, where a procedure—introduced by Leibbrandt and co-workers—called *split-dimensional regularization* is needed in order to render the energy integrals well-defined.

In this paper, we studied Feynman loop integrals pertaining to three outstanding gauges: the usual, and more popular, covariant Feynman gauge and two of the trickiest noncovariant gauges, the light-cone and the Coulomb ones. Our results are given in terms of hypergeometric functions and in the dimensional regularization context.

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